December 1, 1999 WOCOMAL Varsity Meet

ROUND I : Definitions

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Let a non-prime positive integer be written as the product of its prime factors. The number is a *Wocomal* number iff the sum of those prime factors is a perfect square. The smallest Wocomal number is 4 because $4 = 2 \times 2$ and $2+2$ is a perfect square. What is the next smallest Wocomal number?

2. We can devise a shorthand notation for large positive integers by letting d_n stand for the occurrence of the digit d, n consecutive times. For example, $1_4 9_5 8_3 6_2$ denotes 11119999988866. Find x, y, and z if $2_x 3_y 5_z + 3_z 5_x 2_y = 5_3 7_2 8_3 5_1 7_3$.

3. Consider a collection of piles of blocks consisting of one pile of 8 blocks, one pile of 5 blocks, and one pile of 2 blocks. Such a collection, C_1, is expressed as (8,5,2) where the piles are listed in non-increasing order. Obtain a new collection, C_2, by *harvesting* C_1, where *harvesting* is defined to mean remove one block from each pile to form a new pile. Thus C_2 = (7, 4, 3, 1). If C_2 is harvested, C_3 = (6, 4, 3, 2). Starting from C_1 = (10, 6, 3), determine C_{100}.

ANSWERS

(1 pt) 1. ________________

(2 pts) 2. x = ______ y = ______ z = ______

(3 pts) 3. ________________________

Algonquin, Auburn, Shrewsbury

December 1, 1999 WOCOMAL Varsity Meet

ROUND II: Algebra 1, open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. If $\frac{(n-1)+n+(n+1)}{n}=\frac{2001+x+2003}{x}$, where $n\neq 0$ and $x\neq 0$, find x.

2. X and Y can run around a one mile loop route in 6 and 10 minutes respectively. If they start at the same place at the same time and go in opposite directions at these constant speeds, how long in minutes after starting will they meet?

3. Solve this inequality for k: $2-3k\leq 2k-3\leq k+1$

ANSWERS

(1 pt) 1. ____________________

(2 pts) 2. ____________________ min.

(3 pts) 3. ____________________

Clinton, St. John's, Worcester Academy

December 1, 1999 WOCOMAL Varsity Meet

ROUND III: Problem solving

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

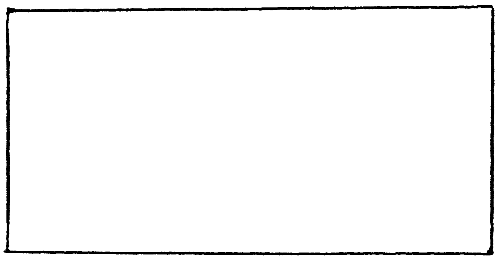
1. Find all whole numbers such that the number increased by the sum of its digits equals 93.

2. A sequence of fractions is Fibonacci if and only if any two successive fractions have the following form.:

$$\ldots, \frac{m}{p}, \frac{p}{m+p}, \ldots$$

If the 9th term of a Fibonacci fraction sequence is $\frac{21}{34}$, find the first term.

3. A square hole is 2 feet on a side. Laura has a rectangular piece of wood 18 inches by 32 inches. How can she cut the rectangle into two pieces involving only right angles and have them fit the square hole exactly? Show how to make the cuts on the rectangle like this in the answer space. Include numbers for dimensions of the pieces!



ANSWERS

(1 pt) 1. ____________

(2 pts) 2. ____________

(3 pts) 3.

Auburn, Shrewsbury, Worcester Academy

December 1, 1999

WOCOMAL Varsity Meet

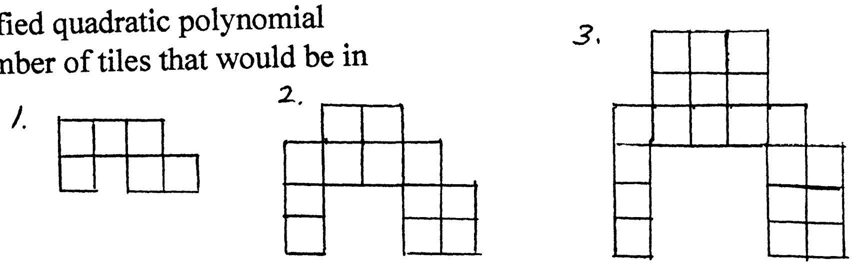
ROUND IV: Sequences and series

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. Find the 18th term of the arithmetic progression $x,\ x + 3a,\ x + 6a, \ldots$

2. The first three terms of a geometric progression are, in order, $x + 1$, $2x$, and $2x + 3$. Find all possible values of x.

3. Consider this tiling pattern with the first three figures shown. Find a simplified quadratic polynomial expression for the number of tiles that would be in the nth figure.



ANSWERS

(1 pt) 1. ______________________

(2 pts) 2. ______________________

(3 pts) 3. ______________________

Auburn, Clinton, Hudson

ROUND V: Matrices

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM

1. If $\begin{bmatrix} 3 & 2 \\ -6 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ 4 \end{bmatrix} = \begin{bmatrix} -7 \\ 34 \end{bmatrix}$, find a.

2. If $A = \begin{bmatrix} 1 & 3 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} x+1 \\ y \end{bmatrix}$, find x and y such that AB = 3B.

3. A general form equation for a circle is $x^2 + y^2 + Dx + Ey + F = 0$, where D, E, and F are constants. If these points, (-2,3), (6, -5), and (0,7), lie on a circle, fill in the missing numbers in the matrix equation like this in the answer section whose solution would be D, E, and F.

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

ANSWERS

(1 pt) 1. ____________

(2 pts) 2. x = ____________ y = ____________

(3 pts) 3. $\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \cdot \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$

Auburn, Holy Name, Westboro

December 1, 1999 WOCOMAL Varsity Meet

TEAM ROUND : Topics of previous rounds and open

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM and ON THE SEPARATE TEAM ANSWER SHEET 2 points each

1. A *lucky* year is one in which at least one date, when written in the form month / day / year, has the following property: the product of the month and the day equals the last two digits of the year. For example, 1956 is lucky because of 7 / 8 / 1956. Which of the ten years 1990-1999 are (is) **not** lucky?

2. For how many values of x is this equation true? $(5+3) \div 4 - x = (5+3) \div (4-x)$

3. How many positive integers less than 100 are the product of 2 or 3 consecutive integers?

4. Find the sum of this infinite seeries in a + bi form. ($i = \sqrt{-1}$)
$$1 + \frac{i}{2} + \frac{i^2}{4} + \frac{i^3}{8} + \frac{i^4}{16} + \ldots$$

5. If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, what is A^{15}?

6. If the vector from the origin to $(-5\sqrt{3}, 5)$ is rotated 120° countercloskwise about the origin, then the resulting vector connects the origin to (a,b). Find the exact coordinates of (a,b).

7. A piece of wire 8 cm long is cut into three pieces so that they form the sides of a triangle. If all three pieces have integer lengths, what is the length of the shortest piece ?

8. If i represents the imaginary unit, find the ordered pair of real numbers (x, y) for which $(2+2i)^9 = x + yi$.

9. Write the seventh term of $\left(x^2 - \frac{1}{x}\right)^9$

Algonquin, Bromfield, Burncoat, Clinton, St. John's, Tahanto, Worcester Academy

December 1, 1999 WOCOMAL Varsity Meet ANSWERS

ROUND I defs

(1 pt) 1. 14

(2 pts) 2. $x = 5$, $y = 4$, $z = 3$

(3 pts) 3. $(6, 5, 3, 3, 1, 1)$ need this order

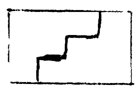
ROUND II alg 1

(1 pt) 1 2002

(2 pts) 2. $3\frac{3}{4}$ or 3.75 min. $\frac{15}{4}$ OK

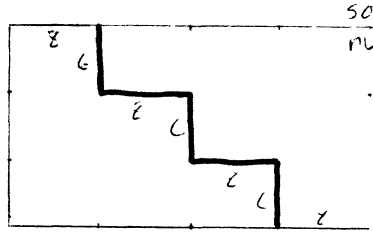
(3 pts) 3. $1 \le K \le 4$

ROUND III prob solv

(1 pt) 1. 78 

(2 pts) 2. 0 or $\frac{0}{1}$ ✓ OK need some numbers!

(3 pts) 3.



ROUND IV seq ser

(1 pt) 1. $x + 51a$

(2 pts) 2. $-\frac{1}{2}$, 3 need both

(3 pts) 3. $n^2 + 3n + 2$

ROUND V matrices

(1 pt) 1. -5

(2 pts) 2. $x = -1$, $y = 0$

(3 pts) 3. $\begin{bmatrix} -2 & 3 & 1 \\ 6 & -5 & 1 \\ 0 & 7 & 1 \end{bmatrix} \begin{bmatrix} D \\ E \\ F \end{bmatrix} = \begin{bmatrix} -13 \\ -61 \\ -49 \end{bmatrix}$

TEAM ROUND 2 pts each

1. 1994, 1997 need both

2. 2 or two

3. 11 or eleven

4. $\frac{4}{5} + \frac{2}{5}i$ or $\frac{4+2i}{5}$

5. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ OR $-A$

6. $(0, -10)$

7. 2 cm

8. $(8192, 8192)$

9. 84

ROUND I

1. Try sum of primes $= 3^2$ and seek smallest possible product. $2+7=9$ and $2\cdot 7 = 14$. Ans. 14

2. Sum is 555778885777.
Work backwards to find that

$$\begin{array}{r} 222223333555 \\ +\ 333555552222 \\ \hline \end{array}$$ works.

$\therefore x = 5,\ y = 4,\ z = 3$

3. $C_1 = (10,6,3) \to C_2 = (9,5,3,2) \to$
$C_3 = (8,4,4,2,1) \to C_4 = (7,5,3,3,1) \to$
$C_5 = (6,5,4,2,2) \to C_6 = (5,5,4,3,1,1) \to$
$C_7 = (6,4,4,3,2) \to C_8 = (5,5,3,3,2,1) \to$
$C_9 = (6,4,4,2,2,1) \to C_{10} = (6,5,3,3,1,1) \to$
$C_{11} = (6,5,4,2,2)$ which $= C_5$ and makes the process periodic starting with C_5 with period 6. $100-5=95$ and $95 \div 6 = 15\frac{5}{6}$.
$\therefore C_{100} = C_{10} = (6,5,3,3,1,1)$

ROUND II

1. By "observation" $x = 2002$ fits the n-pattern
or $\dfrac{3\cancel{n}}{\cancel{n}} = \dfrac{4004+x}{x}$
$3x = 4004 + x \Rightarrow x = 2002$

2. Let t = time until they meet by together completing 1 lap.
$\frac{1}{6}t + \frac{1}{10}t = 1$
$5t + 3t = 30 \Rightarrow t = \frac{30}{8} = 3\frac{3}{4}$ min

3.

$2-3k \le 2k-3$	$2k-3 \le k+1$
$5 \le 5k$	$k \le 4$
$1 \le k$	

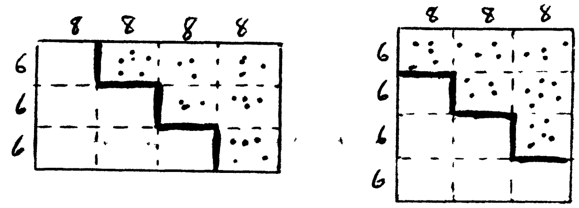
Combine as $1 \le k \le 4$

ROUND III

1. Digit sum ≤ 18, so number ≥ 75 and < 93. Try them. Ans. 78

2. Work backwards
$\frac{21}{34}, \frac{13}{21}, \frac{8}{13}, \frac{5}{8}, \frac{3}{5}, \frac{2}{3}, \frac{1}{2}, \frac{1}{1}, \frac{0}{1}$

3. Area $= 18\cdot 32 = 576 = 24^2$ so need 24×24 square. Work with 6 and 8.



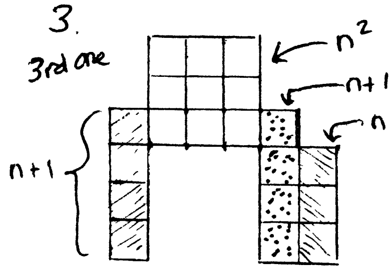
OR 180° rotation

ROUND IV

1. $d = 3a$
$t_{18} = t_1 + 17d$
$= x + 17\cdot 3a = x + 51a$

2. $\dfrac{x+1}{2x} = \dfrac{2x}{2x+3}$ because of const. ratio
$2x^2 + 5x + 3 = 4x^2$
$0 = 2x^2 - 5x - 3 = (2x+1)(x-3)$
$x = -\frac{1}{2}$ or 3

3. 3rd one



check this on $n = 2, 1$

Polynomial
$n^2 + 2(n+1) + n$
$= n^2 + 3n + 2$

ROUND V

1. $3a+8=-7 \Rightarrow a=-5$

   Check that $(-6)(-5)+4$ does $=34$

2. $$\begin{bmatrix} x+1+3y \\ 4x+4+3y \end{bmatrix} = \begin{bmatrix} 3x+3 \\ 3y \end{bmatrix}$$

   Equating elements gets

   $$\left\{\begin{matrix} 3y = 2x+2 \\ 4x+4 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x=-1 \\ y=0 \end{matrix}$$

3. Using pts in circle equation gets

   $$\left\{\begin{matrix} 4+9-2D+3E+F = 0 \\ 36+25+6D-5E+F = 0 \\ 0+49+0+7E+F = 0 \end{matrix}\right.$$

   Rearrange as

   $$\begin{matrix} -2D+3E+F = -13 \\ 6D-5E+F = -61 \\ 0D+7E+F = -49 \end{matrix}$$

   Put numbers in the matrices

TEAM ROUND

1. examples making lucky

   10/9/90 5/19/95
   7/13/91 4/24/96
   4/23/92 7/14/98
   3/31/93 9/11/99

   $94 = 2\cdot 47$
   97 is prime
   $\therefore$ 1994 and 1997

2. $2-x = \frac{8}{4-x}$

   $8-6x+x^2 = 8$

   $x(x-6)=0$

   $x=0$ or 6 ans 2

3. $1\cdot 2 = 2$
   $2\cdot 3 = 6 \leftarrow$
   $3\cdot 4 = 12$
   $4\cdot 5 = 20$
   $5\cdot 6 = 30$
   $6\cdot 7 = 42$
   $7\cdot 8 = 56$
   $8\cdot 9 = 72$
   $9\cdot 10 = 90$
   then too big

   $1\cdot 2\cdot 3 = 6$, but
   $2\cdot 3\cdot 4 = 24$
   $3\cdot 4\cdot 5 = 60$
   $4\cdot 5\cdot 6$ too big

   ans 11

TEAM ROUND cont.

4. $$S_\infty = \frac{a}{1-r} = \frac{1}{1-\frac{i}{2}} = \left(\frac{2}{2-i}\right)\left(\frac{2+i}{2+i}\right)$$

   $$= \frac{4+2i}{4-i^2} = \frac{4}{5} + \frac{2}{5}i$$

5. Easy with suitable calculator, but

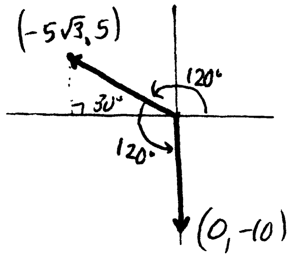
   $A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$

   $A^3 = -A$

   $A^4 = I$

   $A^5 = A$, periodic, period 4

   $A_{15} = A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

6. 

   length

   $= \sqrt{(-5\sqrt{3})^2+5^2}$

   $= \sqrt{75+25}$

   $= 10$

7. Sum of 3 pos integers to be 8.

   For sum of any two > third, cannot use 1.

   2, 3, 3 works. Ans 2

8. $(2+2i)^9 = 2^9(1+i)^9$

   $= 512\left(\left((1+i)^2\right)^4(1+i)\right)$

   $= 512\,(2i)^4(1+i)$

   $= 8192(1+i)$

   Ans (8192, 8192)

9. $$\frac{9\cdot 8\cdot 7}{1\cdot 2\cdot 3}\left(x^2\right)^3\left(-\frac{1}{x}\right)^6$$

   $$= 84\, x^6\left(\frac{1}{x^6}\right)$$

   $$= 84$$